2D Magnetostatic Field Representations on a Rectangular Region

Jean-François Ostiguy
Accelerator Division / Accelerator Physics
MS 345, X2231

February 22, 1991

1 Introduction

The field of accelerator magnets is usually measured using rotating coils. These coils are known as "harmonic probes" because they provide a direct measurement of the coefficients of the harmonic expansion of the magnetic field. In its simplest version, a harmonic probe is simply a long and narrow rectangular coil. When such a coil is set to rotate uniformly with an angular velocity ω_0 in a time-independent 2D magnetic field, a time-dependent periodic voltage is induced. The magnitude and phase of the field harmonics can be extracted from the signal with a spectrum analyzer. In practice, for example in measuring the field of a dipole magnet, the accuracy of the measure is limited by the fact that the signal associated with high order harmonics represents a very small fraction of the fundamental. Modern probes are therefore complex objects where sets of coils are connected together so as to maximize the sensitivity to certain harmonics while minimizing the effect of mechanical assembly errors.

The harmonic expansion of the magnetic field diverges when the distance R between the point of interest and the expansion point (x_0, y_0) is larger than the shortest distance between (x_0, y_0) and the <u>nearest</u> source. Conventional iron-dominated dipole magnets often have a rectangular physical aperture which is wider in the radial (bending) than in the vertical plane. Assuming that the expansion is made about the beam axis, the harmonic series will converge only for $\sqrt{(x^2 + y^2)} < L_y/2$ where L_y is the total vertical aperture. In is important to note that in theory, knowledge of the field harmonics about <u>any</u> point in the aperture is sufficient to characterize the field <u>everywhere</u>. However, for a point located outside the circle of convergence of the series, an indirect method must be used to calculate the field. In relation with the latter statement, it is important to note that the theory refers to a situation where <u>all</u> harmonics are known with <u>infinite</u> accuracy.

In practice, multipoles are measured at two (or more) locations in addition to the origin. This is illustrated in figure 1. By using the appropriate series, the field can be determined over most of the aperture and, if the circle of convergence overlap, on the entire extend of the horizontal axis. Typically, ten to twenty multipoles coefficients are recorded for each location of the axis of the probe. Intuitively, it is clear that the three sets of data obtained are not independent [1]. In fact, one can establish a relation between the three different sets of data and use it to improve the the accuracy on a given coefficient.

Most accelerator tracking codes require the field to be represented by a polynomial in the transverse coordinates x and y. Since its coefficients are directly measurable, the harmonic polynomial is a natural choice. However, **it is important to realize that this polynomial will not correctly predict the field when the beam excursions in the radial plane go beyond the radius of convergence of the harmonic series.**

There are a number of different ways to characterize a 2D field over a rectangular aperture. The most trivial one is to simply store the values of the field at every point on a sufficiently fine rectangular grid. The field at any point (x, y) can then be obtained by interpolation. It is obvious that this approach is not very convenient in practice because it involves a large number of parameters. The object of this note is to document a number of methods which may be used to represent a 2D magnetic field everywhere on a rectangular aperture.

We begin with a brief review of basic 2D magnetostatics with an attempt to emphasize certain important points often treated in a superficial manner in standard textbooks. Integral representations are obtained using both Green's theorem and its complex analog, the Cauchy integral theorem. The relation between these representations is established. Series expansions are obtained for both the Dirichlet and Neumann problems by solving Laplace's equation in rectangular coordinates. Various 2D expansions are obtained by analytically continuing one dimensional expansions. Finally the problem of constructing a polynomial representation valid over a whole rectangular region is discussed.

2 Theoretical Background

2.1 2D Magnetostatics

The equations of magnetostatics in a source-free region

$$\nabla \times \mathbf{B} = 0 \tag{1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

[1] Once again, knowledge of *all* multipoles at *any one* of the three locations is, in theory, sufficient to completely characterize the field everywhere.

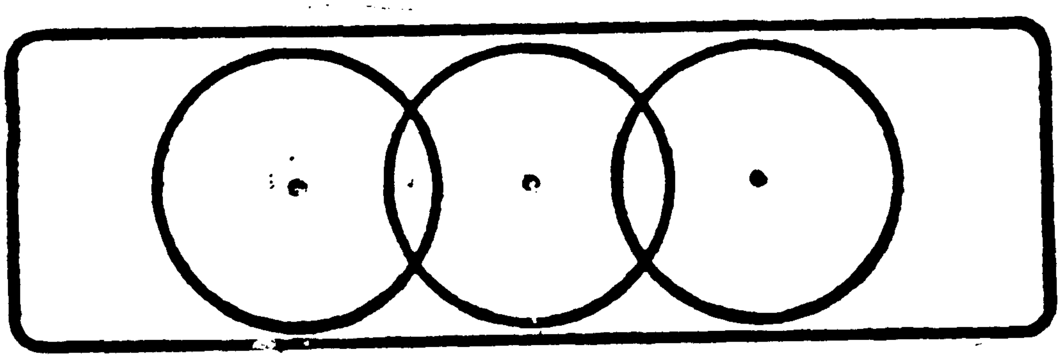

Figure 1: Multipoles are measured at two other locations, in addition to the origin.

become, in 2D cartesian coordinates

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{3}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \tag{4}$$

Assuming Ω is a simply connected region, the magnetic field can be represented by a scalar potential V [2]

$$\mathbf{B} = -\hat{\mathbf{x}}\frac{\partial V}{\partial x} - \hat{\mathbf{y}}\frac{\partial V}{\partial y} \tag{5}$$

or a by a vector potential [3] A

$$\mathbf{B} = +\hat{\mathbf{x}}\frac{\partial A}{\partial y} - \hat{\mathbf{y}}\frac{\partial A}{\partial x} \tag{6}$$

Note that specifying the scalar potential V along a path Γ amounts to specify the tangential component of the magnetic field along Γ. Similarly, specifying A along Γ amounts to specify the normal component of $\mathbf{B}$.

2.2 2D Harmonics

In polar coordinates, Laplace's equation takes on the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = 0 \tag{7}$$

Assuming a solution of the form

$$A(r,\theta) = R(r)\Theta(\theta) \tag{8}$$

and substituting in (7) yields

$$\frac{\Theta''}{\Theta} = k^2 \tag{9}$$

and

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = k^2 \tag{10}$$

where k^2 is a constant. The solutions to equation (10) are

$$R(r) = \begin{cases} 1, \log r & k = 0 \\ r^k, r^{-k} & k \neq 0 \end{cases} \tag{11}$$

Similarly, the solution of equation (9) is

$$\Theta(\theta) = A_n \sin k\theta + B_n \cos k\theta \tag{12}$$

Two separate cases must be considered:

[2] In the MKS system, V is usually defined such that $\mathbf{B} = -\mu_0 \nabla V$.

[3] The terminology "vector" potential refers to the 3D case. In 2D, A is a pseudo scalar.

- The interior solution: Ω is the inside of a circle of radius r_0
- The exterior solution: Ω is the outside of a circle of radius r_0

In the case where $\{\Omega : 0 \leq r < r_0\}$, the solutions $\log r$ and r^{-k} must be rejected due to their singular behavior at the origin. Since the potential cannot be continuous unless Θ has a period 2π, the potential is

$$A(r,\theta) = \sum_{n=0}^{\infty} r^n \left(A_n \sin n\theta + B_n \cos n\theta\right) \tag{13}$$

In the case where $\{\Omega : r_0 < r < \infty\}$, the solutions 1 and r^k must be rejected and one has

$$A(r,\theta) = \sum_{n=0}^{\infty} r^{-n} \left(A_n \sin n\theta + B_n \cos n\theta\right) \tag{14}$$

The solutions (13) and (14) are often expressed in the normalized form

$$A(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^{\pm n} \left(a_n \sin n\theta - b_n \cos n\theta\right) \tag{15}$$

2.3 Complex Potential

It is a well known fact that both the real and imaginary parts of an analytic function $F(z)$ are harmonic functions. This suggests that either V or A could be considered as the real (imaginary) part of an analytic function. A convenient way to represent a 2D magnetic field is to introduce the complex potential

$$F(z) = -(A + iV) \tag{16}$$

Under this definition, $F(z)$ is **analytic** [4]; however, it is important to realize that even though the real and imaginary parts of an analytic function are necessarily harmonic, the converse is not true in general. In other words, a function of the complex variable may have harmonic real and imaginary parts and not be analytic. A necessary and sufficient condition for analyticity is for the Cauchy-Riemann equations to be satisfied. With the definition (16), the two components of the magnetic field can be obtained from the complex potential by differentiating the latter with respect to z

$$\frac{dF}{dz} = B_y + iB_x \tag{17}$$

Interestingly, this result implies that both B_x and B_y are also harmonic functions since all the derivatives of an analytic function are analytic.

[4] Note that $V + iA$ is *not* an analytic function.

Any function analytic in the neighborhood of a point z_0 can be expanded in Taylor series around z_0. Without loss of generality, one can always choose z_0 as the origin and write

$$F(z) = \sum_{n=0}^{\infty} c_n \left(\frac{z}{r_0}\right)^n \tag{18}$$

where

$$c_n = \frac{1}{n!}\frac{d^n F}{d(z/r_0)^n} \tag{19}$$

Writing z in polar form

$$z^n = r^n(\cos n\theta + i \sin n\theta) \tag{20}$$

and setting $c_n = b_n + ia_n$

$$A + iV = \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n [(a_n \sin n\theta - b_n \cos n\theta) - i(b_n \sin n\theta + a_n \cos n\theta)] \tag{21}$$

The coefficients a_n and b_n are simply the Fourier coefficients of the expansion of the potentials around a circle of radius r_0:

$$a_n = +\frac{1}{\pi}\int_0^{2\pi} r_0^n A(r_0, \phi) \sin n\phi \, d\phi \tag{22}$$

$$= \begin{cases} -\frac{1}{2\pi}\int_0^{2\pi} V(r_0, \phi)\, d\phi & n = 0 \\ -\frac{1}{\pi}\int_0^{2\pi} r_0^n V(r_0, \phi) \cos n\phi \, d\phi & n > 0 \end{cases} \tag{23}$$

$$b_n = -\frac{1}{\pi}\int_0^{2\pi} r_0^n V(r_0, \phi) \sin n\phi \, d\phi \tag{24}$$

$$= \begin{cases} -\frac{1}{2\pi}\int_0^{2\pi} A(r_0, \phi)\, d\phi & n = 0 \\ -\frac{1}{\pi}\int_0^{2\pi} r_0^n A(r_0, \phi) \cos n\phi \, d\phi & n > 0 \end{cases} \tag{25}$$

2.4 Analytic Continuation

An analytic function is a remarkable mathematical object. It is defined to be a function (of the complex variable) with a unique first derivative throughout a region. A function so defined has extraordinary properties. It turns out to have unique derivatives of all orders. Its real and imaginary parts are harmonic. Its line integral is independent of the path. The values of the function at points on a close curve Γ determine its value at points inside Γ. The principle of analytic continuation is the ultimate fabulous property of analytic functions. According to this principle, an analytic function is uniquely determined everywhere in the complex plane by its values in any neighborhood, however small, of a point z_0. In fact, the values of the function along a path segment, however short suffice to uniquely determine the function everywhere. In this case, however, one must be given that the function is analytic in the region including the line segment. In more precise terms, the following theorem holds:

Identity theorem for analytic functions

If two functions are analytic in a region Ω, and they coincide in a neighborhood, however small of a point z_0 of Ω or only along a path segment, however small, terminating in z_0, then the two functions are equal everywhere in Ω.

This implies that an analytic continuation, if it exists at all, is unique. Suppose one has an analytic function $f_1(z)$ defined in a region Ω_1. Suppose Ω_2 is another region which overlaps Ω_1. If there exists a function $f_2(z)$ which is analytic on Ω_2 and equal to $f_1(z)$ in the region $\Omega_1 \cap \Omega_2$, then $f_2(z)$ is unique. In fact, f_1 and f_2 are just partial representatives of a function $f(z)$ analytic throughout the combined region $\Omega_1 \cup \Omega_2$. The so-called circle-chain method employing power series provides a method which **in principle** can be used to effect the analytic continuation. Start with a function defined by a power series in one circle. Use the values of the function so obtained to make a power series expansion about a point inside but near the edge of the circle, etc, etc. A consequence of the identity theorem for analytic functions, the analytic continuation obtained in this way is unique. The processes can be pushed as far as possible by extending the circle-chain in all directions the radii of the circles being, of course, the radii of convergence of the power series. Since whenever the function is analytic it can be expanded into a power series, these circles eventually reach to every nook and cranny of the complex plane where it is possible to analytically continue the function. Although it is not possible to analytically continue through a singularity, but it will, in most cases be possible to analytically continue around it. Note that an analytic continuation **cannot** be carried through if one meets a continuous line of singularities separating one part of the complex plane from the rest.

2.5 Multipole "Feeddown"

Consider a function $G(z)$ analytic on a region Ω. One can obtain a relation between the coefficients c_n of the Taylor series expansion

$$G(z) = \sum_{n=0}^{\infty} c_n z^n \tag{26}$$

about the origin and the coefficients d_n of the expansion about another point z_0 by using the following trick:

$$G(z) = \sum_{n=0}^{\infty} c_n (z - z_0 + z_0)^n \tag{27}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} c_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k \tag{28}$$

where the binomial theorem has been used to expand $(z - z_0 + z_0)^n$ and it is assumed that $|z| < |z_0|$. It is clear from the above result that the term of order n in the expansion about the origin contributes to the terms of order $1, 2 \cdots, n$ in the expansion about z_0. This phenomenon is sometimes called "multipole feeddown" since for example, a pure quadrupole field at the origin would result into a field at z_0 which has both a dipole and a quadrupole component. A relation between c_n and d_n can be obtained by rearranging the terms in (28). Let us consider the coefficient affecting the term in $(z - z_0)^n$. Clearly, the contributions to this coefficient come from the terms

$$\begin{array}{c} c_n \binom{n}{n} z_0 \\ c_{n+1} \binom{n+1}{n} z_0^{(n+1)-n} \\ c_{n+2} \binom{n+2}{n} z_0^{(n+2)-n} \\ \vdots \end{array} \tag{29}$$

Equation (28) can obviously be written in the form

$$G(z) = \sum_{n=0}^{\infty} \left[\sum_{k=n}^{\infty} c_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^n \tag{30}$$

By identification with

$$G(z) = \sum_{n=0}^{\infty} d_n (z - z_0)^n \tag{31}$$

one sees that

$$d_n = \sum_{k=n}^{\infty} c_n \binom{k}{n} z_0^{k-n} \tag{32}$$

It is important to point out that **this relation is valid only if the point z_0 lies inside the circle of convergence of the series (26).**

3 Representations

In this section, we shall obtain different expressions for the magnetic field in a rectangular region. In all cases, the field (potential) will be expressed as a linear combination of basis functions $\psi_k(x, y)$ [5].

$$A(x, y) = \sum_{k=0}^{\infty} c_k \psi_k(x, y) \tag{33}$$

[5]We consider here the discrete case, but the sum can be continuous.

Each basis function can be expanded in a harmonic series of the form

$$\psi_k(r,\theta) = \sum_{k=0} \left(\frac{r}{r_0}\right)^n (a_{nk} \sin n\theta - b_{nk} \cos n\theta) \tag{34}$$

where

$$a_{nk} = +\frac{1}{\pi} \int_0^{2\pi} r_0^n \psi_k(r_0,\phi) \sin n\phi \, d\phi \tag{35}$$

$$b_{nk} = \begin{cases} -\frac{1}{2\pi} \int_0^{2\pi} \psi_k(r_0,\phi)\, d\phi & n = 0 \\ -\frac{1}{\pi} \int_0^{1\pi} r_0^n \psi_k(r_0,\phi) \cos n\phi \, d\phi & n > 0 \end{cases} \tag{36}$$

The above relations enable us to express the coefficients of the multipole expansion of A in terms of c_k

$$a_n = \sum_{k=0}^{\infty} a_{nk} c_k \tag{37}$$

$$b_n = \sum_{k=0}^{\infty} b_{nk} c_k \tag{38}$$

Written in matrix form (37) and (38) become

$$\mathbf{a} = A\mathbf{c} \tag{39}$$

$$\mathbf{b} = B\mathbf{c} \tag{40}$$

In general, the matrices A and B may be singular [6]. To solve for the coefficient c_k one can form the weighted sum of (39) and (40)

$$\mathbf{a} + \lambda\mathbf{b} = [A + \lambda B]\mathbf{c} \tag{41}$$

where λ is a suitably chosen constant [7]. In general, the truncated version of this (infinite) system will not have no solution. Nevertheless, one can obtain a solution in the sense of least squares

$$\mathbf{c} = \left[[A+\lambda B]^\dagger [A+\lambda B]\right]^{-1} [A+\lambda B]^\dagger [\mathbf{a}+\mathbf{b}] \tag{42}$$

In the case where the field (potential) is a complex function, the multipole expansion takes on the form

$$F(z) = \sum_{k=0}^{\infty} d_k \psi_k(z) \tag{43}$$

[6] Consider, for example, the case where the source geometry is such that all skew multipoles a_n vanish. The matrix A is obviously singular in that case.

[7] More generally one could substitute a positive definite matrix for λ.

The basis functions $\psi_k(z)$ are now functions of the complex variable. Expanding $\psi_k(z)$ around the origin

$$\psi_k(z) = \sum_{n=0}^{\infty} c_{nk} z^n \tag{44}$$

But one can also expand $F(z)$

$$F(z) = \sum_{k=0}^{\infty} c_k z^k \tag{45}$$

Substituting (44) into (43) and comparing with (45)

$$c_k = \sum_{n=0}^{\infty} c_{nk} d_k \tag{46}$$

A truncated version of this system of equations can be solved for c_k in the sense of least squares.

3.1 Integral Representations (Method of Sources)

3.1.1 Integral Representations based on Green's Identities

Consider an arbitrarily shaped, simply connected region Ω. We shall denote a point (x, y) by $\mathbf{r}$ and a point (x', y') by $\mathbf{r}'$. Let the distance R between $\mathbf{r}$ and $\mathbf{r}'$ be defined as

$$R \equiv |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2} \tag{47}$$

If $\psi(\mathbf{r}')$ and $\phi(\mathbf{r}')$ are two arbitrary scalar functions defined on Ω, the following identity, known as Green's identity, holds

$$\int_{\Omega} \phi \nabla'^2 \psi - \psi \nabla'^2 \phi \, dS' = \int_{\partial\Omega} \psi(\hat{\mathbf{n}} \cdot \nabla' \phi) - \phi(\mathbf{n} \cdot \nabla' \psi) \, ds' \tag{48}$$

where the primes on the Laplacian operator and the differential elements mean that the derivatives and integrations are to be performed with respect to primed coordinates. Consider the function

$$\Phi(\mathbf{r}; \mathbf{r}') = -\frac{1}{2\pi} \log\left(\frac{1}{R}\right) = \frac{1}{2\pi} \log R \tag{49}$$

It is a simple matter to verify that Φ is harmonic for all $\mathbf{r}' \neq \mathbf{r}$. More specifically, it can be shown that

$$\nabla'^2 \Phi = \nabla^2 \Phi = \delta(\mathbf{r} - \mathbf{r}') \tag{50}$$

Setting $\phi = \Phi$ and $\psi = V(\mathbf{r}')$ in (44) one obtains, for any $\mathbf{r} \in \Omega$

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{2\pi} \left[\int_{\partial\Omega} V \hat{\mathbf{n}} \cdot \nabla' \log\left(\frac{1}{R}\right) - \log\left(\frac{1}{R}\right) (\hat{\mathbf{n}} \cdot \nabla' V) \right] ds' & (51) \\ &= \frac{1}{2\pi} \left[\int_{\partial\Omega} \vec{\sigma}_p \cdot \nabla' \log\left(\frac{1}{R}\right) - \sigma \log\left(\frac{1}{R}\right) \right] ds' & (52) \end{aligned}$$

The scalar potential V inside the region Ω can be interpreted as the resultant of the potentials produced by two kinds of sources:

- a surface pole distribution of density $\sigma = \hat{\mathbf{n}} \cdot \nabla V$
- a surface dipole distribution of density $\vec{\sigma}_p = \hat{\mathbf{n}} V$

These sources may be regarded as fictitious sources accounting for the influence of the sources situated in the region exterior to Ω. It is important to note that they cannot be used to determine the field in the exterior region. On the contrary, these sources make the potential vanish in the exterior region since one has, for $\mathbf{r} \notin \Omega$

$$\frac{1}{2\pi} \int_{\partial\Omega} \vec{\sigma}_p \cdot \nabla' \log\left(\frac{1}{R}\right) - \sigma \log\left(\frac{1}{R}\right) \, ds = 0 \tag{53}$$

This result is another demonstration of the fact that the normal and tangential components of the field (or equivalently, the real and imaginary parts of the complex potential) cannot be independently specified at all points of $\partial\Omega$. If one considers the potential as a known quantity and the fictitious source distributions as unknowns, equation (52) can in principle be used to determine σ and $\vec{\sigma}_p$. Unfortunately, σ and $\vec{\sigma}_p$ are not independent quantities and the solution is unique only if (53) is enforced simultaneously.

It turns out that it is also possible to represent the field by a distribution constituted **exclusively** of poles or dipoles. Furthermore, no auxiliary constraint similar to (53) need to be enforced for the distributions to be unique. Consider the situation where the potential to be represented is harmonic in the region **exterior** to Ω (in other words the sources of V are situated inside Ω). Applying Green's identity and assuming that the **exterior potential** $\bar{V}$ vanishes at infinity

$$\bar{V}(\mathbf{r}) = \frac{1}{2\pi} \int_{\partial\Omega} \bar{V} \hat{\mathbf{n}} \cdot \nabla \log\left(\frac{1}{R}\right) - \log\left(\frac{1}{R}\right) (\hat{\mathbf{n}} \cdot \nabla \bar{V}) \, ds' \tag{54}$$

for $\mathbf{r}$ in the exterior region and

$$\bar{V}(\mathbf{r}) = 0 \tag{55}$$

for $\mathbf{r}$ in the interior region. Adding (52) and (54) yields for the interior region

$$V(\mathbf{r}) = \frac{1}{2\pi} \int_{\partial\Omega} (V - \bar{V}) \hat{\mathbf{n}} \cdot \nabla' \log\left(\frac{1}{R}\right) - \log\left(\frac{1}{R}\right) (\hat{\mathbf{n}} \cdot \nabla' V - \hat{\mathbf{n}} \cdot \nabla' \bar{V})) \, ds' \tag{56}$$

where $\hat{\mathbf{n}}$ is a normal unit vector pointing outside Ω If one chooses $\bar{V}$ such that

$$V - \bar{V} = 0 \text{ on } \partial\Omega \tag{57}$$

(56) becomes

$$V(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial\Omega} -\log\left(\frac{1}{R}\right)(\hat{\mathbf{n}}\cdot\nabla(V-\bar{V}))\,ds \tag{58}$$

which can be put in the form

$$V(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial\Omega} \sigma(\mathbf{r}')\log R\,ds' \tag{59}$$

Thus, any function V harmonic within Ω can be represented by a distribution of fictitious poles on $\partial\Omega$. Note that since V and $\bar{V}$ are unique, so is σ. In the same manner, one can also choose $\bar{V}$ such that

$$\hat{\mathbf{n}}\cdot\nabla'V - \hat{\mathbf{n}}\cdot\nabla'\bar{V} = 0 \tag{60}$$

(56) then becomes

$$V(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial\Omega}(V-\bar{V})\,\hat{\mathbf{n}}\cdot\nabla'\log\left(\frac{1}{R}\right)\,ds' \tag{61}$$

which can be put in the form

$$V(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial\Omega}\sigma_p\hat{\mathbf{n}}\cdot\nabla'\log R\,ds' \tag{62}$$

i.e., any function harmonic in Ω can be represented by a unique distribution of fictitious dipoles on $\partial\Omega$. Since the vector potential A is harmonic in a source-free region, the representations (59) and (62) can also be used for A.

At this point, the connection between relations (59), (62) and the general form (33) may not be clear. This connection is perhaps easier to establish when (59) and (62) are written in discrete form. For example, relation (59) becomes

$$V(x,y) = \sum_k \sigma(s_k)\,\Delta_k\frac{1}{2\pi}\log\sqrt{(x-x')^2+(y-y')^2} \tag{63}$$

Here, $s_k = (x'_k, y'_k)$ represents a point on the boundary at the center of an interval of width Δ_k. Clearly, the basis functions in the expansion (33) are

$$\psi_k(x,y) = \frac{1}{2\pi}\log\sqrt{(x-x')^2+(y-y')^2} \tag{64}$$

and the coefficients c_k

$$c_k = \sigma(s_k)\Delta_k \tag{65}$$

3.1.2 Cauchy Integral Theorem

A most important result from the theory of analytic functions, Cauchy's integral theorem, states that the values of an analytic function everywhere inside a simply connected region Ω are completely determined by its values on the boundary. Cauchy's integral theorem provides a method to actually compute the scalar potential $F(z)$ for $z \in \Omega$ given $F(z)$ on $\partial\Omega$.

$$F(z) = \frac{1}{2\pi i} \oint \frac{F(t)}{t-z} dt \tag{66}$$

Interestingly, the real and imaginary parts of a analytic function on a closed path are not independent since one must have

$$\frac{1}{2\pi i} \oint \frac{F(t)}{t-z} dt = 0 \tag{67}$$

for $z \notin \Omega$. In fact, one can demonstrate that the values of the complex potential (or any other analytic function) $F(z) = -A - iV$ inside a region Ω are completely determined by either its real or imaginary values on $\partial\Omega$. Let

$$\begin{aligned}
z &\equiv x + iy && (68)\\
t &\equiv x' + iy' && (69)\\
X &\equiv x - x' && (70)\\
Y &\equiv y - y' && (71)\\
R &\equiv \sqrt{(x-x')^2 + (y-y')^2} && (72)
\end{aligned}$$

With the above definitions, (66) becomes

$$\begin{aligned}
F(z) &= -\frac{1}{2\pi i} \int \left[\frac{A+iV}{t-z}\right] \left[\frac{(t-z)^*}{(t-z)^*}\right] dt && (73)\\
&= \frac{1}{2\pi} \int \left[\frac{A+iV}{R}\right] \left[\frac{Y-iX}{R}\right] (dx' + idy') && (74)\\
&= \frac{1}{2\pi} \int \left[\frac{A+iV}{R}\right] \left[\frac{(Ydx' - Xdy') + i(Xdx' + Ydy')}{R}\right] && (75)
\end{aligned}$$

But

$$\begin{aligned}
\frac{Xdy' - Ydx'}{R} &= \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} && (76)\\
\frac{Xdx' + Ydy'}{R} &= \hat{\mathbf{r}} \cdot \hat{\mathbf{s}} && (77)
\end{aligned}$$

where $\hat{\mathbf{n}}$ is the outward normal unit vector. Now, consider the real part of (75).

$$A = \frac{1}{2\pi} \int \frac{A}{R} \hat{\mathbf{r}} \cdot \mathbf{n} + \frac{V}{R} \hat{\mathbf{r}} \cdot \mathbf{s} \, ds' \tag{78}$$

Integrating by parts,

$$\int \frac{V}{R}\hat{\mathbf{r}} \cdot \hat{s}\, ds' = -\log\left(\frac{1}{R}\right) \hat{\mathbf{n}} \cdot \nabla' A \tag{79}$$

where the identity

$$\hat{\mathbf{s}} \cdot \nabla' V = \hat{\mathbf{n}} \cdot \nabla' A \tag{80}$$

has been used. The first term of the integrand in equation (78) can be put in the form

$$\hat{\mathbf{n}} \cdot \frac{\hat{\mathbf{r}}}{R} = \hat{\mathbf{n}} \cdot \nabla' \left(\frac{1}{R}\right) \tag{81}$$

Finally

$$A(\mathbf{r}) = \frac{1}{2\pi} \int_{\partial\Omega} A\, \hat{\mathbf{n}} \cdot \nabla' \left(\frac{1}{R}\right) - \log\left(\frac{1}{R}\right) \hat{\mathbf{n}} \cdot \nabla' A\, ds' \tag{82}$$

A similar result can be obtained for V. The Cauchy formula can therefore be viewed as a compact way of expressing (52). However, it also has the significant advantage that the constraint (53) is automatically enforced. Thus, given the complex potential $F(z)$ inside the aperture, one would write

$$F(z) \simeq \sum_n \frac{1}{2\pi i} \frac{1}{(t_k - z)} F(t_k) \Delta_k \tag{83}$$

$$= \sum_n \frac{1}{2\pi i} \frac{1}{(t_k - z)} c_k \tag{84}$$

and solve for the unknown (complex) coefficients c_k.

3.2 Series Expansions

Any function harmonic on a rectangular region can be expressed in terms of its values and that of its derivative on the boundary by solving Laplace's equation directly by the method of separation of variables. The coefficients of the series are then the coefficients of Fourier series expansions of the boundary data. In the present case, both magnetic potentials satisfy Laplace's equation i.e.,

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0 \tag{85}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{86}$$

For the purpose of this note, we shall deal exclusively with A. Similar expressions can be obtained for V with little effort.

$$A(x, y) = X(x)Y(y) \tag{87}$$

Substituting (87) into (85) yields the two ordinary differential equations

$$\frac{d^2X}{dx^2} = \pm k^2 X \tag{88}$$

$$\frac{d^2Y}{dy^2} = \mp k^2 Y \tag{89}$$

where k is real and positive. In what follows, we shall consider the rectangular region illustrated in figure 2. For convenience, we introduce the following notation

$$\xi \equiv \left[\frac{x - x_{\min}}{x_{\max} - x_{\min}}\right] \tag{90}$$

$$\eta \equiv \left[\frac{y - y_{\min}}{y_{\max} - y_{\min}}\right] \tag{91}$$

$$L_x \equiv (x_{\max} - x_{\min}) \tag{92}$$

$$L_y \equiv (y_{\max} - y_{\min}) \tag{93}$$

$$\alpha \equiv L_x / L_y \tag{94}$$

3.2.1 The Dirichlet Problem

We first consider the so-called Dirichlet problem whose solution expresses the potential inside a closed region in terms of its values on the boundary. Thus, we assume that

$$\begin{aligned} A(x,y) &= f_1(\eta) \quad \xi = 0, 0 < \eta < 1 \\ A(x,y) &= f_2(\eta) \quad \xi = 1, 0 < \eta < 1 \\ A(x,y) &= f_3(\xi) \quad \eta = 0, 0 < \xi < 1 \\ A(x,y) &= f_4(\xi) \quad \eta = 0, 0 < \xi < 1 \end{aligned} \tag{95}$$

Depending on the sign chosen for the separation constant, one gets either

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L_x}(x - x_{\min}) + \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{L_x}(x - x_{\min}) \tag{96}$$

$$Y(y) = \sum_{n=1}^{\infty} C_n \cosh \frac{n\pi}{L_x}(y - y_{\min}) + D_n \sinh \frac{n\pi}{L_x}(y - y_{\min}) \tag{97}$$

or

$$X(x) = \sum_{n=1}^{\infty} C_n \cosh \frac{n\pi}{L_y}(x - x_{\min}) + D_n \sinh \frac{n\pi}{L_y}(x - x_{\min}) \tag{98}$$

$$Y(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L_y}(y - y_{\min}) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L_y}(y - y_{\min}) \tag{99}$$

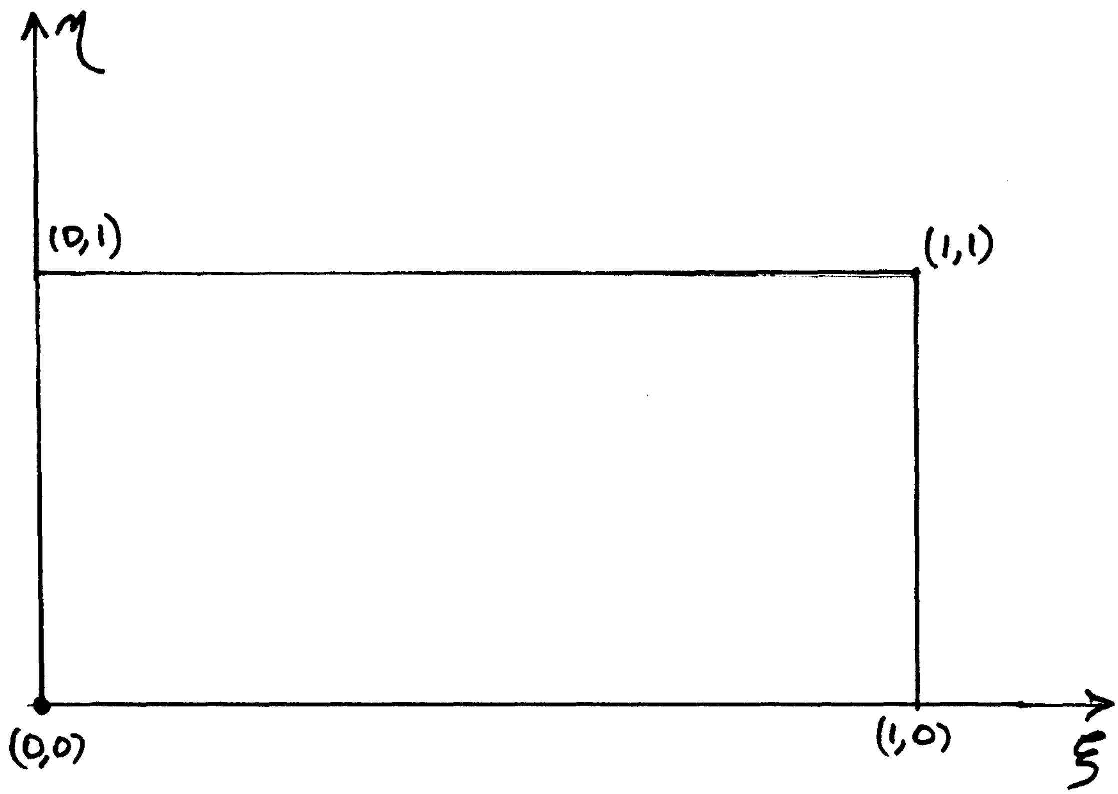

Figure 2: Coordinate system for section 3. 2.

The most general solution to the Dirichlet problem is a linear combination of the above solutions over a range of values of the separation constant. Rather than attempting to satisfy all the boundary conditions at once, it is more practical to exploit the linearity of Laplace's equation and consider the auxiliary problem

$$\begin{aligned} A(x,y) &= f_1(\eta) \quad \xi = 0, 0 < \eta < 1 \\ A(x,y) &= 0 \qquad \xi = 1, 0 < \eta < 1 \\ A(x,y) &= 0 \qquad \eta = 0, 0 < \xi < 1 \\ A(x,y) &= 0 \qquad \eta = 1, 0 < \xi < 1 \end{aligned} \tag{100}$$

Once the solution to the auxiliary problem is known, the general solution can be obtained trivially by superposition.

Since the potential has to vanish at $\eta = 0$ and $\eta = 1$, one can assume a solution of the form

$$A(x,y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi\alpha\xi + B_n \sinh n\pi\alpha\xi) \sin n\pi\eta \tag{101}$$

for (100). By imposing the boundary condition at $\xi = 0$, it is immediately apparent that the coefficients A_n are simply the coefficients of the Fourier sine series expansion of $f_1(\eta)$. For the potential to vanish at $\xi = 1$ one must have

$$A_n \cosh n\pi\alpha + B_n \sinh n\pi\alpha = 0 \tag{102}$$

Using this relation and expressing the sum of hyperbolic functions as as hyperbolic sine of the sum of their respective arguments one gets

$$A(x,y) = \sum_{n=1}^{\infty} A_n \left[\frac{\sinh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha} \right] \sin n\pi\eta \tag{103}$$

The solution to the general problem is therefore

$$A(x,y) = A_1(x,y) + A_2(x,y) + A_3(x,y) + A_4(x,y) \tag{104}$$

where

$$A_1(x,y) = \sum_{n=1}^{\infty} A_n \left[\frac{\sinh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha} \right] \sin n\pi\eta \tag{105}$$

$$A_2(x,y) = \sum_{n=1}^{\infty} B_n \left[\frac{\sinh n\pi\alpha\xi}{\sinh n\pi\alpha} \right] \sin n\pi\eta \tag{106}$$

$$A_3(x,y) = \sum_{n=1}^{\infty} C_n \left[\frac{\sinh n\pi(1-\eta)/\alpha}{\sinh n\pi/\alpha} \right] \sin n\pi\xi \tag{107}$$

$$A_4(x,y) = \sum_{n=1}^{\infty} D_n \left[\frac{\sinh n\pi\eta/\alpha}{\sinh n\pi/\alpha} \right] \sin n\pi\xi \tag{108}$$

and

$$
\begin{aligned}
A_n &= 2\int_0^1 f_1(\eta)\sin n\pi\eta \; d\eta && (109)\\
B_n &= 2\int_0^1 f_2(\eta)\sin n\pi\eta \; d\eta && (110)\\
C_n &= 2\int_0^1 f_3(\xi)\sin n\pi\xi \; d\xi && (111)\\
D_n &= 2\int_0^1 f_4(\xi)\sin n\pi\xi \; d\xi && (112)
\end{aligned}
$$

Since the tangential derivative of the vector potential determines uniquely the normal magnetic field, one can also write series in terms of the Fourier coefficients of the normal components of the field

$$
\begin{aligned}
A_1(x,y) &= \sum_{n=1}^{\infty} \frac{L_y}{n\pi} E_n \left[\frac{\sinh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha}\right] \sin n\pi\eta && (113)\\
A_2(x,y) &= \sum_{n=1}^{\infty} \frac{L_y}{n\pi} F_n \left[\frac{\sinh n\pi\alpha\xi}{\sinh n\pi\alpha}\right] \sin n\pi\eta && (114)\\
A_3(x,y) &= \sum_{n=1}^{\infty} -\frac{L_x}{n\pi} G_n \left[\frac{\sinh n\pi(1-\eta)/\alpha}{\sinh n\pi/\alpha}\right] \sin n\pi\xi && (115)\\
A_4(x,y) &= \sum_{n=1}^{\infty} -\frac{L_x}{n\pi} H_n \left[\frac{\sinh n\pi\eta/\alpha}{\sinh n\pi/\alpha}\right] \sin n\pi\xi && (116)
\end{aligned}
$$

and

$$
\begin{aligned}
E_n &= 2\int_0^1 B_x(0,\eta)\sin n\pi\eta \; d\eta && (117)\\
F_n &= 2\int_0^1 B_x(1,\eta)\sin n\pi\eta \; d\eta && (118)\\
G_n &= 2\int_0^1 B_y(\xi,0)\sin n\pi\xi \; d\xi && (119)\\
H_n &= 2\int_0^1 B_y(\xi,1)\sin n\pi\xi \; d\xi && (120)
\end{aligned}
$$

3.2.2 The Neumann Problem

The solution to the Neumann problem expresses the potential inside a closed region in terms of the values of its normal derivative on the boundary. The Neumann problem for the vector potential is defined by the following boundary conditions:

$$
\begin{array}{lllll}
-\partial A/\partial x & = B_y(0,y) & = g_1(\eta) & \xi = 0, 0 < \eta < 1 \\
-\partial A/\partial x & = B_y(L_x,y) & = g_2(\eta) & \xi = 1, 0 < \eta < 1 \\
\partial A/\partial y & = B_x(x,0) & = g_3(\xi) & \eta = 0, 0 < \xi < 1 \\
\partial A/\partial y & = B_x(x,L_y) & = g_4(\xi) & \eta = 0, 0 < \xi < 1
\end{array}
\tag{121}
$$

According to a well-known theorem of potential theory, the solution to this problem is unique within an arbitrary constant.

In contrast with the Dirichlet problem, the boundary data for the Neumann problem must satisfy a compatibility condition [8]. This is merely a consequence of the requirement that the data be compatible with the integral form of Maxwell's equations i.e

$$\oint \mathbf{B} \cdot d\mathbf{l} = 0 \tag{122}$$

The compatibility condition makes it necessary to use a different strategy to solve the general Neumann problem. We first consider the following auxiliary problem

$$
\begin{array}{lllll}
-\partial A^{(1)}/\partial x & = B_y(0,y) & = \bar{g}_1(\eta) & \xi = 0, 0 < \eta < 1 \\
-\partial A^{(1)}/\partial x & = B_y(L_x,y) & = 0 & \xi = 1, 0 < \eta < 1 \\
\partial A^{(1)}/\partial y & = B_x(x,0) & = 0 & \eta = 0, 0 < \xi < 1 \\
\partial A^{(1)}/\partial y & = B_x(x,L_y) & = 0 & \eta = 1, 0 < \xi < 1
\end{array}
\tag{123}
$$

where

$$\bar{g}_1(\eta) \equiv g_1(\eta) - A_0 \tag{124}$$

$$A_0 \equiv \int_0^1 g_1(\eta)\, d\eta \tag{125}$$

Clearly,

$$\int_0^1 \bar{g}_1(\eta)\, d\eta = 0 \tag{126}$$

and the boundary data for this auxiliary problem is compatible. By a procedure similar to the one used to solve the Dirichlet problem, it is easily shown that

$$A^{(1)}(x,y) = \sum_{n=1}^{\infty} \frac{L_x}{n\pi\alpha} A_n \left[\frac{\cosh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha}\right] \cos n\pi\eta \tag{127}$$

Similarly, using superposition, the solution to the problem defined by

$$
\begin{array}{lllll}
-\partial \bar{A}/\partial x & = B_y(0,y) & = \bar{g}_1(\eta) & \xi = 0, 0 < \eta < 1 \\
-\partial \bar{A}/\partial x & = B_y(L_x,y) & = \bar{g}_2(\eta) & \xi = 1, 0 < \eta < 1 \\
+\partial \bar{A}/\partial y & = B_x(x,0) & = \bar{g}_3(\xi) & \eta = 0, 0 < \xi < 1 \\
+\partial \bar{A}/\partial y & = B_x(x,L_y) & = \bar{g}_4(\xi) & \eta = 1, 0 < \xi < 1
\end{array}
\tag{128}
$$

[8] In fact, continuity of the potential on the boundary ensures that the data for the Dirichlet problem is compatible with the integral form of Maxwell's equations.

is [9]

$$\bar{A}(x,y) = A^{(1)}(x,y) + A^{(2)}(x,y) + A^{(3)}(x,y) + A^{(4)}(x,y) \tag{129}$$

where

$$A^{(1)}(x,y) = +\frac{L_x}{n\pi\alpha}\sum_{n=1}^{\infty} A_n \left[\frac{\cosh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha}\right]\cos n\pi\eta \tag{130}$$

$$A^{(2)}2(x,y) = -\frac{L_x}{n\pi\alpha}\sum_{n=1}^{\infty} B_n \left[\frac{\cosh n\pi\alpha(1-\xi)}{\sinh n\pi\alpha}\right]\cos n\pi\eta \tag{131}$$

$$A^{(3)}(x,y) = -\frac{L_y}{n\pi/\alpha}\sum_{n=1}^{\infty} C_n \left[\frac{\cosh n\pi(1-\eta)/\alpha}{\sinh n\pi/\alpha}\right]\cos n\pi\xi \tag{132}$$

$$A^{(4)}(x,y) = +\frac{L_y}{n\pi/\alpha}\sum_{n=1}^{\infty} D_n \left[\frac{\cosh n\pi\eta/\alpha}{\sinh n\pi/\alpha}\right]\cos n\pi\xi \tag{133}$$

and

$$A_n = 2\int_0^1 g_1(\eta)\cos n\pi\eta \, d\eta \tag{134}$$

$$B_n = 2\int_0^1 g_2(\eta)\cos n\pi\eta \, d\eta \tag{135}$$

$$C_n = 2\int_0^1 g_3(\xi)\cos n\pi\xi \, d\xi \tag{136}$$

$$D_n = 2\int_0^1 g_4(\xi)\cos n\pi\xi \, d\xi \tag{137}$$

(129) is obviously not the solution to the general Neumann problem. The latter can be obtained by adding the solution to the following problem to $\bar{A}$ as defined by (123)

$$\begin{array}{lllll} -\partial A^{(0)}/\partial x & = B_y(0,y) & = A_0 & \xi = 0, 0 < \eta < 1 \\ -\partial A^{(0)}/\partial x & = B_y(L_x,y) & = B_0 & \xi = 1, 0 < \eta < 1 \\ \partial A^{(0)}/\partial y & = B_x(x,0) & = C_0 & \eta = 0, 0 < \xi < 1 \\ \partial A^{(0)}/\partial y & = B_x(x,L_y) & = D_0 & \eta = 0, 0 < \xi < 1 \end{array} \tag{138}$$

where the compatibility condition

$$-A_0 L_y + B_0 L_y + C_0 L_x - D_0 L_x = 0 \tag{139}$$

holds. Interestingly, it is not possible to solve the boundary value problem defined by (121) directly using separation of variables. However, (121) can

[9] $\bar{g}_2, \bar{g}_3, \bar{g}_4$ are defined in a similar way as $\bar{g}_1$ i.e., by subtracting the average values B_0, C_0, D_0 of g_2, g_3 and g_4.

be converted into a mathematically equivalent Dirichlet problem for the scalar potential $V^{(0)}$. Setting $V^{(0)}(0,0) = 0$, one has

$$\begin{aligned}
V^{(0)}(0,y) &= -C_0L_x - B_0L_y + D_0L_x + A_0(y-L_y) && \xi = 0, 0 < \eta < 1 \\
V^{(0)}(L_x,y) &= -C_0L_x - B_0y && \xi = 1, 0 < \eta < 1 \\
V^{(0)}(x,0) &= -C_0x && \eta = 0, 0 < \xi < 1 \\
V^{(0)}(x,L_y) &= -C_0L_x - B_0L_y + D_0(x-L_x) && \eta = 0, 0 < \xi < 1
\end{aligned} \tag{140}$$

In principle this problem could be solved using the technique which we already used to solve the general Dirichlet problem. However, a a function which satisfies the conditions (140) can be found by inspection. To see how this is done, consider the function

$$\phi(x,y) = K_1x + K_2y + K_3xy + K_4 \tag{141}$$

$\phi(x,y)$ is obviously harmonic; furthermore, it degenerates into a linear function of the axial coordinate along each one of the boundaries. Clearly, the problem is solved if one can find K_1, K_2, K_3 and K_4 such that

$$\phi(0,0) = V^{(0)}(0,0) \tag{142}$$

$$\phi(L_x,0) = V^{(0)}(L_x,0) \tag{143}$$

$$\phi(L_x,L_y) = V^{(0)}(L_x,L_y) \tag{144}$$

$$\phi(0,L_y) = V^{(0)}(0,L_y) \tag{145}$$

It is easily verified that

$$\phi(x,y) = -A_0y - C_0x + \left[\frac{A_0 - B_0}{L_x}\right]xy \tag{146}$$

The corresponding vector potential can then be determined by integrating the field.

$$A^{(0)}(x,y) = C_0y - A_0x + \left[\frac{A_0 - B_0}{2L_x}\right](x^2 - y^2) \tag{147}$$

Thus, the general solution to the Neumann problem (121) is

$$A(x,y) = A^{(0)}(x,y) + A^{(1)}(x,y) + A^{(2)}(x,y) + A^{(3)}(x,y) + A^{(4)}(x,y) \tag{148}$$

Note that since the normal derivative of the potential is specified on the boundary the solution expresses the potential inside the region in terms of the Fourier coefficients of the tangential field on the boundary.

3.2.3 Alternate Series

In the two previous sections, we have established series which express the vector potential A as a function of its values and those of its tangential derivative on the boundary. It it possible to construct other series by using other types of boundary conditions. One popular choice is to fix the value of A and its tangential derivative on the horizontal symmetry axis which is usually chosen to coincide with the x-axis.

$$\begin{aligned} A(x,y) = \sum_{n=0}^{\infty} \quad & A_n \sin nk_x x \sinh nk_x y + B_n \sin nk_x x \cosh nk_x y \\ + \; & C_n \cos nk_x x \sinh nk_x y + D_n \cos nk_x x \cosh nk_x y \end{aligned} \tag{149}$$

where

$$k_x \equiv \frac{2\pi}{L_x} \tag{150}$$

$$A_n = +\frac{1}{n\pi}\int_{-L_x/2}^{L_x/2} B_x(x,0) \sin nk_x x \, dx \tag{151}$$

$$B_n = \begin{cases} -\frac{1}{2\pi}\int_{-L_x/2}^{L_x/2} B_y(x,0)\, dx & n=0 \\ -\frac{1}{n\pi}\int_{-L_x/2}^{L_x/2} B_y(x,0)\cos nk_x x\, dx & n\neq 0 \end{cases} \tag{152}$$

$$C_n = \begin{cases} +\frac{1}{2\pi}\int_{-L_x/2}^{L_x/2} B_x(x,0)\, dx & n=0 \\ +\frac{1}{n\pi}\int_{-L_x/2}^{L_x/2} B_x(x,0)\cos nk_x x\, dx & n\neq 0 \end{cases} \tag{153}$$

$$D_n = +\frac{1}{n\pi}\int_{-L_x/2}^{L_x/2} B_y(x,0) \sin nk_x x \, dx \tag{154}$$

3.3 Series Obtained by Analytic Continuation

A powerful way to obtain a series representation of an analytic function over a region Ω is to obtain a series which converges on a line segment included in Ω and to extend this series over the whole plane using the principle of analytic continuation. In particular, if the segment is chosen so as to coincide with either the real or the complex axis, the arsenal of method used for real approximations becomes available to construct approximations.

3.3.1 General Fourier Series Expansion

We shall consider here the system of coordinates shown in figure 3. Along the x-axis, any analytic function $\tilde{B}$ can be seen as a complex function of the real variable x. More specifically [10] , if

$$\tilde{B}(z) = B_y(z) + iB_x(z) \tag{155}$$

[10] In this section, the notation $\tilde{B}$ is used to avoid confusion between the real magnetic field and the complex magnetic field.

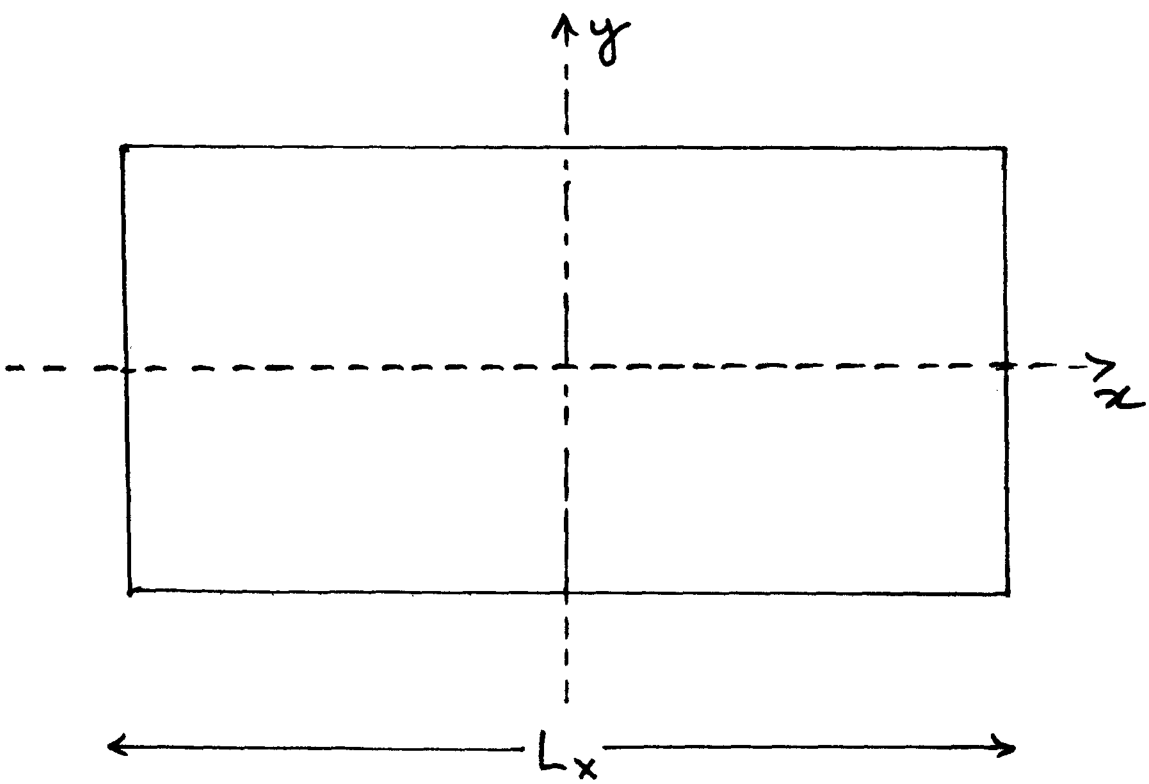

Figure 3: Coordinate system for section 3. 3.

then, along x, one can write

$$\tilde{B}(x) = B_y(x) + iB_x(x) \tag{156}$$

Seen as a function of x, $\tilde{B}$ can be expanded as a Fourier series

$$\tilde{B}(x) = \sum_{n=-\infty}^{\infty} d_n e^{ink_x x} \tag{157}$$

$$= \sum_{n=-\infty}^{\infty} [c_n + i\bar{c}_n] e^{ink_x x} \tag{158}$$

where

$$c_n \equiv \frac{1}{L} \int_{-L_x/2}^{L_x/2} B_y(x) e^{-ink_x x}\, dx \tag{159}$$

$$\bar{c}_n \equiv \frac{1}{L} \int_{-L_x/2}^{L_x/2} B_x(x) e^{-ink_x x}\, dx \tag{160}$$

Let

$$c_n = b_n + ia_n \tag{161}$$

$$\bar{c}_n = \bar{b}_n + i\bar{a}_n \tag{162}$$

Equation (158) becomes

$$\begin{aligned} \tilde{B}(x) = (b_0 + i\bar{b}_0) &+ \textstyle\sum_{n=1}^{\infty} (b_n + ia_n) e^{ink_x x} + (b_n - ia_n) e^{-ink_x x} \\ &+i \textstyle\sum_{n=1}^{\infty} (\bar{b}_n + i\bar{a}_n) e^{ink_x x} + (\bar{b}_n - i\bar{a}_n) e^{-ink_x x} \end{aligned} \tag{163}$$

$$\begin{aligned} \tilde{B}(x) = (b_0 + i\bar{b}_0) &+ \textstyle\sum_{n=1}^{\infty} (b_n + i\bar{b}_n) \left[e^{ink_x x} + e^{-ink_x x}\right] \\ &+i \textstyle\sum_{n=1}^{\infty} (\bar{b}_n + i\bar{a}_n) \left[e^{ink_x x} + e^{-ink_x x}\right] \end{aligned} \tag{164}$$

Collecting terms

$$\tilde{B}(x) = (b_0 + i\bar{b}_0) + \sum_{n=1}^{\infty} 2(b_n + i\bar{b}_n) \cos nk_x x - 2(a_n + i\bar{a}_n) \sin nk_x x \tag{165}$$

Substituting $z = x + iy$ for x and using the identities

$$\sin nkz = \sin nk_x x \cosh nk_x y + i \cos nk_x x \sinh nk_x y \tag{166}$$

$$\cos nkz = \cos nk_x x \cosh nk_x y - i \sin nk_x x \sinh nk_x y \tag{167}$$

equation (165) becomes

$$\begin{aligned} B(z) = &\textstyle\sum_{n=1}^{\infty} \quad 2\left(b_n \cos nk_x x \cosh nk_x y - a_n \sin nk_x x \cosh nk_x y\right) \\ &\qquad + 2\left(\bar{b}_n \sin nk_x x \sinh nk_x y + \bar{a}_n \cos nk_x x \sinh nk_x y\right) \\ +i &\textstyle\sum_{n=1}^{\infty} - 2\left(b_n \sin nk_x x \sinh nk_x y - a_n \cos nk_x x \sinh nk_x y\right) \\ &\qquad + 2\left(\bar{b}_n \cos nk_x x \cosh nk_x y - \bar{a}_n \sin nk_x x \cosh nk_x y\right) \end{aligned} \tag{168}$$

This result is identical to (149).

3.3.2 Cosine Series

A different series can be obtained by expanding the field as a cosine series.

$$\tilde{B}(x) = \sum_{n=0}^{\infty}(c_n + i\bar{c}_n)\cos 2nk_x x \tag{169}$$

$$\begin{aligned} B(z) \quad = \quad & \textstyle\sum_{n=0}^{\infty} \left(c_n \cos \frac{n}{2}k_x x \cosh \frac{n}{2}k_x y + \bar{c}_n \sin \frac{n}{2}nk_x x \sinh \frac{n}{2}nk_x y\right) \\ +i\ & \textstyle\sum_{n=0}^{\infty} \left(\bar{c}_n \cos \frac{n}{2}k_x x \cosh \frac{n}{2}k_x y - c_n \sin \frac{n}{2}k_x x \sinh \frac{n}{2}k_x y\right) \end{aligned} \tag{170}$$

where

$$c_0 = \frac{1}{L_x}\int_{-L_x/2}^{L_x/2} \tilde{B}\, dx \tag{171}$$

$$c_n + i\bar{c}_n = \frac{2}{L_x}\int_{-L_x/2}^{L_x/2} \tilde{B}\cos\frac{n}{2}k_x x\, dx \tag{172}$$

3.3.3 Sine series

One can also expand the complex field $\tilde{B}$ as a sine series along x.

$$\tilde{B}(x) = \sum_{n=1}^{\infty}(c_n + i\bar{c}_n)\sin\frac{n}{2}k_x x \tag{173}$$

Substituting $z = x + iy$ for x in (154) yields

$$\begin{aligned} B(z) \quad = \quad & \textstyle\sum_{n=1}^{\infty} \left(c_n \sin \frac{n}{2}k_x x \cosh \frac{n}{2}k_x y - \bar{c}_n \cos \frac{n}{2}k_x x \sinh \frac{n}{2}k_x y\right) \\ +i\ & \textstyle\sum_{n=1}^{\infty} \left(\bar{c}_n \sin \frac{n}{2}k_x x \cosh \frac{n}{2}k_x y + c_n \cos \frac{n}{2}k_x x \sinh \frac{n}{2}k_x y\right) \end{aligned} \tag{174}$$

where

$$c_n + i\bar{c}_n = \frac{2}{L_x}\int_{-L_x/2}^{L_x/2} \tilde{B}\sin\frac{n}{2}k_x x\, dx \tag{175}$$

4 L^2 Polynomial Approximations

We wish to approximate a function $f(z)$ analytic over a region Ω by a series of the form

$$f(z) \simeq \sum_{n=0}^{N} c_n p_n(z) \tag{176}$$

$\{z^n\}$ in the case of complex functions defined over a convex simply connected region Ω [2]. The importance of this result is that it is **theoretically** possible to approximate to any degree of accuracy an analytic function by a polynomial (in the sense of the least square norm). In other words, $\epsilon(N)$ can be made arbitrarily small by increasing the maximum order N.

The elements of the matrix P are easily calculated

$$\langle p_m(z)|p_n(z)\rangle = \int_\Omega (z^m)^* z^n \, dxdy \tag{184}$$

$$= \int_\Omega (x-iy)^m (x+iy)^n \, dxdy \tag{185}$$

In the particular case where f is a real function and Ω is a finite interval on the real axis one gets, with $p_n(x) = x^n$

$$\langle p_m|p_n\rangle = \int_{-L_x/2}^{L_x/2} x^{m+n} \, dx \tag{186}$$

$$= (-1)^{m+n+1} \frac{L_x^{m+n+1}}{(m+n+1)} \tag{187}$$

This matrix, known as the Hilbert matrix, is notoriously ill-conditioned. This difficulty can be eliminated by performing a Gram-Schmidt orthogonalization of the polynomial base. The matrix P then becomes diagonal if the othogonal polynomials are used as a the expansion basis. It turns out that for real functions, the orthogonal polynomials are the Legendre polynomials. In the complex case, the situation is a lot more complicated because the coefficients of the orthogonal polynomials now depend on the shape of the region Ω. It is interesting to note that when the region Ω is a unit disk the set $\{z^n\}$ is already orthogonal. This is easily be demonstrated by writing z^n in polar form:

$$\int (z^m)^* z^n dxdy = \int_0^{2\pi} \int_0^1 r^{n+m} e^{i(n-m)\phi} \, d\phi dr \tag{188}$$

$$= \frac{\pi}{n+m+1} \delta_{nm} \tag{189}$$

Using Gram-Schmidt orthonormalization algorithm, one can construct a set of orthogonal polynomials $\{\Pi_n(z)\}$ on a rectangle of width L and of height αL

$$\Pi_n(z) = z^n + a_{n-1}(\alpha, L) z^{n-1} + a_{n-2}(\alpha, L) z^{n-2} + \ldots a_0(\alpha, L) \tag{190}$$

Since the orthogonalization process ultimately involves only integration of polynomials of the form $x^n y^m$ over a rectangular region, the coefficients $a_n(\alpha, L)$ can be expressed analytically and tabulated. One then has

$$f(z) \simeq \sum_{n=0}^{N} c_n \Pi_n(z) \tag{191}$$

Taking the inner product on each side,

$$c_n = \frac{1}{\langle \Pi_n(z) | \Pi_n(z) \rangle} \langle \Pi_n(z) | f(z) \rangle \tag{192}$$

This solves the problem of the best approximation in the least square norm.

5 Comments and Conclusion

To the extent that tracking codes expect a polynomial representation of the field as an input, the L_2 approximation of the field over the whole rectangular aperture seems to be the most appropriate representation to solve the problems caused by the divergence of the standard multipole expansion. This may indeed be the case; however, two important points must be kept in mind

- The theory guarantees that the error can be made as small as one wishes. However, this may require the use of very high order polynomials. This fact sets a lower bound on the error since high order polynomials will quickly create overflow problems in a computer.
- The L_2 approximation will oscillate slightly around the exact field. This may be the source of non physical instabilities.

One of the statements made by L. Michelotti in his paper concerning the reconstruction of the field in a rectangle aperture from the multipole data is erroneous. The functions z^n do indeed form a complete basis over a rectangle [2]. It is well known that any simply connected convex domain can be conformally mapped onto the unit disk. Since the set $\{z^n\}$ constitutes an orthogonal basis on the unit disk, the functions defined by applying the mapping to $\{z^n\}$ will be mapped into an orthogonal basis on the rectangle. The mapped basis functions obtained in this manner are obviously not polynomials. This orthogonal basis is, however, not unique.

The results presented in this note are theoretical. The usefulness of the expansions will have to be investigated by performing numerical experiments. The result of these experiments will be communicated in a upcoming note.

6 Acknowledgments

The author would like to thank Leo Michelotti for his valuable comments and for pointing out to him the problem posed by the divergence of the multipole series in tracking codes.

References

[1] L. Michelotti, "*Combining Mutipole Data*"
1987 IEEE Particle Accelerator Conference
Washington D.C. March 16-19
pp. 1645-47

[2] See for example A.I. Markushevich, "*Theory of Functions of a Complex Variable*", Chelsea Publishing Co (1965)